Today + next time : Quotients $G \backslash X$

1 further lecture on ECs

Factorization of isogenies:

$\phi : E \to E_1$, $\psi : E \to E_2$

s.th. $\ker \phi \leq \ker \psi$

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[dr, "\psi"'] & E_1 \arrow[d, dashed, "\exists !"] \\ & E_2 \end{tikzcd}$$

(Do yourself : Extend proof from §11)

Cor Given isogeny $\phi : E \to E'$,

E' uniquely det by $\ker \phi$ (up to ~~isogeny~~ isom)

2 Questions

1) Describe E' in terms of $\ker \phi$

2) Given $K \subseteq E$ fin subgroup,

$\exists ?\ \phi$ w/ $\ker \phi = K$?

Answer $E' = E / \ker(\phi)$, E/K exists.

§1 Group actions (Reference: van-der-Geer – Moonen §IV.)

S base, G/S grp sch

X/S any

Def Action $\underset{\text{def}}{=}$ $G \times_S X \xrightarrow{\mu} X$ s.th.

$\forall T/S$ $\mu(T): G(T) \times X(T) \longrightarrow X(T)$ is set-theoretic grp action.

$$\Leftrightarrow \begin{tikzcd} G \times G \times X \arrow[r, "id \times \mu"] \arrow[d, "m \times id"'] & G \times X \arrow[d, "\mu"] \\ G \times X \arrow[r, "\mu"] & X \end{tikzcd} \quad \& \quad \begin{tikzcd} X \arrow[r, "e \times id"] \arrow[dr, equal] & G \times X \arrow[d, "\mu"] \\ & X \end{tikzcd}$$

Examples 1) Γ grp

$\Gamma_S = \coprod_{\gamma \in \Gamma} S$ - action on X

$=$ Γ-action on X $:=$ $\Gamma \longrightarrow \mathrm{Aut}(X/S)$

2) $\mathbb{G}_{m,S} = \underline{\mathrm{Spec}}\, \mathcal{O}_S[t, t^{-1}] \curvearrowright \mathbb{A}^1_S$ w/ coord x

$$\mathbb{G}_{m,S} \times_S \mathbb{A}^1_S \xrightarrow{\mu} \mathbb{A}^1_S, \quad \mu^*(x) = t \otimes x$$

$\Leftrightarrow$ (on T-val points) $\quad (t, x) \mapsto tx$.

E.g. $\mu_{n,S} := \mathbb{G}_{m,S}[n]$

restricts to $\mu_{n,S} \times_S \mathbb{A}^1_S \longrightarrow \mathbb{A}^1_S$.

3) $GL_{n+1,S} \times_S \mathbb{P}^n_S \longrightarrow \mathbb{P}^n_S$ etc...

Def 1) $U \subseteq X$ open <u>G-stable</u> $\overset{=}{\text{def}}$ $\mu^{-1}(U) = p^{-1}(U)$

($p: G \times_S X \longrightarrow X$ projection)

U G-stable $\Rightarrow$ μ restricts to G-action on U.

2) $$\mathcal{O}_X(X)^G := \mathrm{Eq}\left(\mathcal{O}_X(X) \overset{\mu^*}{\underset{p^*}{\rightrightarrows}} \mathcal{O}_{G \times_S X}(G \times_S X) \right)$$

G-<u>invariants</u>

§2 Quotients

S, G, X, μ as before

a) Categorical quotient $\overset{=}{\text{def}}$

G-invariant $X \xrightarrow{\pi} Y$ s.th.

$$\begin{tikzcd} X \arrow[r, "\pi"] \arrow[d, "f"] & Y \arrow[dl, dashed, "\exists!"] \\ Z & \end{tikzcd}$$

$\forall$ G-invariant f

Unique up to unique iso

f <u>G-invariant</u> $\overset{=}{\text{def}}$ $f \circ \mu = f \circ p$

<u>Question</u> Does a categorical quot exist?

b) <u>Geometric quotient</u>

$$Y := |G| \backslash |X| := \text{Quotient}\left(|G \times_S X| \overset{|\mu|}{\underset{|p|}{\rightrightarrows}} |X| \right)$$

$:=$ top space $|X| \xrightarrow{\pi} Y$ s.th.

$$\begin{tikzcd} {|X|} \arrow[r, "\pi"] \arrow[d, "f"] & Y \arrow[dl, dashed, "\exists!"] \\ Z & \end{tikzcd}$$

$\forall$ cont maps f

s.th. $f \circ |\mu| = f \circ |p|$

Invariant functions $\mathcal{O}_Y := (\pi_* \mathcal{O}_X)^G$.

$\Big(\forall\, U \subseteq Y$ open, $\pi^{-1} U \subseteq X$ is open + G-stable,

$$\mathcal{O}_Y(U) := \mathcal{O}_X(\pi^{-1} U)^G \ . \Big)$$

$(Y, \mathcal{O}_Y)$ is ringed space.

<u>Question:</u> Is $(Y, \mathcal{O}_Y)$ a scheme & π morph of schemes?

c) h_X / h_G quotient in fpqc-sheaves.

$$h_X / h_G : Sch/S^{op} \longrightarrow Set$$

<u>Question:</u> Is this functor representable?

d) $[G \backslash X]$ stacky quotient.

<u>Questions</u> What are its properties?

For $K \subseteq E$ subgroup, all answers are "yes"

+ constructions coincide.

§3 Quotients of affines.

From now on: $S = \operatorname{Spec} R$ affine

$X = \operatorname{Spec} A$

$G = \operatorname{Spec} H / S$ finite loc free.

$\mu : G \times_S X \to X$ as before

$$A^G := \{ a \in A \mid \mu^*(a) = \underset{\substack{\| \\ 1 \otimes a}}{p^*(a)} \text{ in } H \otimes_R A \}$$

<u>Thm</u> $\operatorname{Spec} A \xrightarrow{\pi} \operatorname{Spec} A^G$ is a categorical & geometric quotient.

<u>Proof</u> Obvious:
$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "\pi"] \arrow[d] & \operatorname{Spec} A^G \arrow[dl, dashed, "\exists !"] \\ \operatorname{Spec} B & \end{tikzcd}$$
$\forall$ G-invariant to affine

By adjunction, just the diagram

$$\begin{tikzcd}
H \otimes_R A & A \arrow[l, shift right] \arrow[l, shift left] & A^G \arrow[l, hook'] \\
 & B \arrow[u] \arrow[ur, dotted, "\exists!"'] &
\end{tikzcd}$$

compositions coincide

Difficulty: Extend to general case.

<u>1st step</u> $A^G \hookrightarrow A$ integral

.) Restrict to open + closed of S $\Longrightarrow$ wlog H of rank r

.) $H \otimes_R A$ proj rk r A-module via μ^*.

$\forall\, a \in A$, $\mu^*(a)$ A-linear endo of $H \otimes_R A$

$\rightsquigarrow$ $\chi(t) = t^r + c_{r-1} t^{r-1} + \dots + c_0 \in A[t]$

char poly.

<u>Claim</u> χ G-invariant, i.e. $c_i \in A^G$.

$$\iff \quad \mu^*(\chi(t)) = p^*(\chi(t))$$

Observe $\varphi(\chi(t))$, $\varphi : A \to B$ any, is char poly of $(id_H \otimes \varphi)(\mu^*(a))$ (Functoriality of det.)

$\Rightarrow$ $\mu^*(\chi(t))$ = char poly of

$$(id \otimes \mu^*)(\mu^*(a)) = (m^* \otimes id_A)(\mu^*(a))$$

$\cap$

$H \otimes_R H \otimes_R A$

But $\exists$ also (non-standard) Cartesian square:

$$\begin{tikzcd}
(m\otimes id)^*(\mu^*(a)) & \mu^*(a) \arrow[l, mapsto, "m\times id"'] \\
G\times G\times X \arrow[r] \arrow[dd, "p_{23}"'] & G\times X \arrow[dd, "p"] \\
p^*\chi & \chi \arrow[l, mapsto] \\
G\times X \arrow[r, "p"'] & X
\end{tikzcd}$$

(Vs $id \times m : G\times G \to G\times G$ is automorphism.)

(Similarity w/ argument $\Omega'_{G/S} \simeq p^* e^* \Omega'_{G/S}$)

$\Rightarrow$ $p^*(\chi(t))$ char poly of $(m \otimes id)^*(\mu^*(a))$

<u>Claim</u> $\chi(a) = 0$, in partic a integral over A^G as claimed.

$$H \otimes_R A \xrightarrow{e^* \otimes \mathrm{id}_A} A$$

$$\mu^*(a) - p^*(a) \longmapsto 0$$

$$\Rightarrow \det\left(\mu^*(a) - a \mid H \otimes_R A \right) = 0$$

$$\Rightarrow \chi(a) = 0. \qquad \square \text{ Step 1.}$$

<u>Step 2</u> (Commutative algebra)

<u>Prop</u> (Going up, Stacks 00GH)

$A \xrightarrow{\varphi} B$ integral, $\mathfrak{p}' \subseteq \mathfrak{p}$ in $\operatorname{Spec} A$

$$\mathfrak{p}' = \varphi^{-1}(\mathfrak{q}') \quad \mathfrak{q}' \in \operatorname{Spec} B$$

Then $\exists\ \mathfrak{q} \in \operatorname{Spec} B$ s.th. $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$.

<u>Cor</u> $\pi : |\operatorname{Spec} A| \to |\operatorname{Spec} A^G|$ is surjective + closed.

Proof $A^G \subseteq A$ is <u>subring</u>, $\forall \mathfrak{p} \subseteq A^G$

$$A^G_{\mathfrak{p}} \subseteq A_{\mathfrak{p}} \neq 0\text{-ring.}$$

$$\Longrightarrow \exists\, \mathfrak{q}' \subseteq A \text{ w/ } A^G \cap \mathfrak{q}' \subseteq \mathfrak{p}$$

$$\overset{\text{Going up}}{\Longrightarrow} \quad \mathfrak{p} \in \operatorname{Im}(\pi).$$

If $Z = V(\mathfrak{a}) \subseteq |\operatorname{Spec} A|$, then

$Z \to \operatorname{Spec} A^G$ factors through

$V(\mathfrak{b} := \mathfrak{a} \cap A^G)$

& $A^G/\mathfrak{b} \hookrightarrow A/\mathfrak{a}$ is integral.

$\Longrightarrow$ Surjectivity just proven shows $\pi(Z) = V(\mathfrak{b})$.

$\square$

<u>Step 3</u> $|\operatorname{Spec} A^G| = |G| \backslash |\operatorname{Spec} A|$

·) Step 2 means that

$$|\operatorname{Spec} A| \xrightarrow{\pi} |\operatorname{Spec} A^G|$$

continuous, surjective + closed.

In prtic, it's quotient of top space.

(i.e $U \subseteq |\operatorname{Spec} A^G|$ open

$\Leftrightarrow \pi^{-1} U$ open.)

Only left to show $|\operatorname{Spec} A^G| = |G| \backslash |\operatorname{Spec} A|$

as sets.

Def $\exists\, q \in |G \times_S X|$ w/ $\mu(q) = p$, $p(q) = p'$

[i.e. $p \sim p'$ in $\operatorname{Spec} A$

$\Leftrightarrow \pi(p) = \pi(p')$

.) One direction is clear:

$$p \sim p' \Longrightarrow \pi(p) = \pi(p')$$

since π G-invariant.

.) Conversely assume $p \cap A^G = p' \cap A^G$

$\{q_1, \dots, q_s\} = p^{-1}(p') \subseteq \operatorname{Spec} H \otimes_R A$

Want i s.t. $\mu(q_i) = p$.

Enough $\mathfrak{p} \subseteq \bigcup_i \mu(\mathfrak{q}_i)$.

Assume $a \in \mathfrak{p} \setminus \bigcup_i \mu(\mathfrak{q}_i)$.

Equivalently, $\mu^*(a)$, invertible in all $\kappa(\mathfrak{q}_i)$

$\Longrightarrow \det_A(\mu^*(a) \mid H \otimes_R A)$ invertible in $\kappa(\mathfrak{p}')$.

But $= c_0$ of $\chi(t)$

and $\chi(a) = 0$, so

$c_0 = \pm a(a^{r-1} + c_{r-1}a^{r-2} + \dots + c_1)$
$\in \mathfrak{p}$

Since χ G-invariant,

even $\det = c_0 \in \mathfrak{p} \cap A^G = \mathfrak{p}' \cap A^G$
$\subseteq \mathfrak{p}'$

so $c_0 \notin \kappa(\mathfrak{p}')^\times$ $\lightning$.

Step 4 π is categorical quotient

$\{$ G-stable $U \subseteq |\mathrm{Spec}\, A|\}$

$\overset{\pi^{-1}}{\cong} \{$ open $V \subseteq |\mathrm{Spec}\, A^G|\}$

Given

$$\begin{array}{c} \mathrm{Spec}\, A \\ u \downarrow \\ Y \end{array}$$

G-invariant

pick $Y = \bigcup_i U_i$ affine open

$\Rightarrow \{u^{-1} U_i\}$ G-stable affine open cover of Spec A.

By Step 3 $u^{-1} U_i = \pi^{-1} V_i$ for $V_i \subseteq \mathrm{Spec}\, A^G$ open.

Refining, wlk $V_i = \bigcup D(f_{ij})$

Since $(A[f^{-1}])^G = A^G[f^{-1}]$ $\forall f \in A^G$,

$$\Rightarrow \begin{tikzcd} \pi^{-1} D(f_{ij}) \arrow[r] \arrow[d, "u"] & D(f_{ij}) \arrow[dl, dashed] \\ U_i & \end{tikzcd}$$

$\exists !$ since U_i affine

Loc. of quotient factorization = Quotient fact. of loc.

Then use uniqueness to glue to map

$$\operatorname{Spec} A^G \longrightarrow Y. \qquad \square \text{ Thm.}$$

<u>Rmk</u> $G = \underline{\Gamma}_S = \coprod_{\gamma \in \Gamma} \operatorname{Spec} R$

<u>Clear</u> $\operatorname{charpoly}\left((a_\gamma)_{\gamma \in \Gamma} \,\middle|\, \bigoplus_{\gamma \in \Gamma} A \,;\, t \right)$

$$= \prod_{\gamma \in \Gamma} (t - a_\gamma)$$

$G \subset \operatorname{Spec} A \iff \Gamma^{op}\text{-action on } A$

Then $\mu^*(a) = (\gamma \cdot a)_{\gamma \in \Gamma}$

$\Longrightarrow \chi(t)$ (as in proof)

$$= \prod_{\gamma \in \Gamma} (t - \gamma \cdot a)$$

<u>Clearly</u> Γ-invariant coefficients

+ $\chi(a) = 0$

+ c_0 "norm" $= \prod_{\gamma \in \Gamma} \gamma \cdot a \in A$

<u>Ex</u> $\{\pm 1\} \circlearrowright \mathbb{A}^1$, $(-1)\cdot x = -x$

Geometric quotient : $\mathbb{A}^1 \longrightarrow \mathbb{A}^1$

$x \longmapsto x^2$

since $\{f(x) \in R[x] \mid f(x) = f(-x)\}$

$2 \in R^\times$ $= \{ f(x^2) \}$.